The University of Chicago

# KNOTTED VARIETIES

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS
1933

By
ANNA ADELAIDE STAFFORD

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# KNOTTED VARIETIES

Introduction. A knotted curve is any simple closed curve in Euclidean 3-space, $S_3$. A representation of such a curve can be secured by twisting the ends of a thread about one another in any arbitrary way, and then fastening the ends together. If the thread, so fastened, can be deformed into a circle by bending and stretching only, it is said to be unknotted. The problem of determining when two arbitrarily given knots are equivalent, that is, when one can be continuously deformed, without severing it, into the shape of the other, is at present not completely solved. However, there exist invariants sufficient to distinguish the type of all knotted curves up to those of nine crossings. The portions of this paper referring to ordinary knots in $S_3$ follow closely the work of Alexander (1),* and Reidemeister (1).

It is obvious that a proper knot, that is, one which is not deformable into a circle, cannot exist in a plane. It will be shown, (it is in fact well known), that in $S_4$ a knot can always be deformed into a circle. Hence $S_3$ is the only space in which a proper knot can exist, whereas the unknotted curve, the circle, can exist in $S_2$.

It is the purpose of this paper to show that there exist simple closed varieties of all dimensions which are not deformable by bending or stretching (isotopic transformations), into circuits or spheres of the same dimension, and also, that such n-dimensional

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*References are indicated by the author's name, followed by a number in parentheses which corresponds to the bibliography at the end of the paper.

knotted varieties exist only in $S_{2n+1}$, except for the degenerate case, the n-dimensional sphere, which exists in $S_{n+1}$. Furthermore, it will be shown that the aforesaid varieties can always be deformed into n-dimensional spheres in $S_{2n+2}$.

In the later part of the paper some of the general theory of knot invariants will be extended to knotted varieties, in particular, the polynomial invariants of Alexander will be developed and shown to be topological invariants of the knotted varieties.

1. Definitions and Transformations. A knot is a simple closed polygon in ordinary Euclidean space of three dimensions. It consists of a finite number of vertices and edges. It is not necessary that these edges be considered straight. They are rather the usual zero- and one-dimensional cells of combinatorial topology, Veblen (1) page 1, and the objects which constitute a cell and its boundary will be referred to as points. A knot is, then, a set of a finite number of 0- and 1-cells satisfying the incidence relations of a one-dimensional circuit, Veblen (1) page 16.

Two knots are equivalent, or of the same type, if and only if they are isotopic, Veblen (1) page 188. For the purposes of this paper, an isotopic transformation consists of a set of a finite number of the following operations, Alexander (1) page 276.

(i) To subdivide a one-cell into two one-cells by introducing a new zero-cell at an interior point of the one-cell.

(ii) To reverse the last operation: i.e., to amalgamate a pair of adjacent one-cells and their common boundary into a single one-cell.

(iii) To change the shape of the knot by continuously displacing a vertex (along with the two edges meeting at the

vertex) in such a manner that the knot never acquires a singularity during the process, or, in combinatorial terms, a zero-cell, $a^0$, together with the two one-cells $a^1$ and $b^1$ with which it is incident, may be replaced by any other zero-cell $b^0$ in the $S_3$ and two one-cells $c^1$ and $d^1$ each having $b^0$ as one boundary point, and the other two boundaries of $a^1$ and $b^1$ for their other boundaries respectively, provided $b^0$ is not already a vertex or on an edge of the knot, and that a two-cell can be found in the $S_3$ in which the knot is immersed which is bounded by $a^1b^1d^1c^1$ and contains no point of the knot.

A singularity is any multiple point, line, etc. In combinatorial terms, a cell which fails at some point, line, etc. to be in (1-1) correspondence with a simplex, and is therefore not properly a cell. A singularity on a circuit consists of some part of the circuit where the cells of which it is composed do not satisfy the incidence relations of a sphere of the same dimension.

The diagram of a knot is the projection of the curve from a conveniently chosen point of $S_3$ into a plane. The diagram will be a closed curve, having, in general, singularities. The latter may all be made into double points with distinct tangents by resorting to one or more of the operations described above. The singularities are the crossing points of the diagram. The regions into which the plane is divided by the diagram are the regions of the diagram. The diagram of a proper knot has necessarily three or more crossing points, (Fig. 2).

A diagram will be called reduced when, after a series of applications of transformations (i) and (ii), there are no zero-cells other than crossing points, and the one-cells have on their boundaries only the crossing points. Then by Euler's formula for

polyhedra, if there are $\nu$ zero-cells, there are $2\nu$ one-cells and $\nu + 2$ two-cells on the diagram.

A link is a figure composed of the zero- and one-cells of a finite number of non-intersecting knots. Two curves will be called linked or sometimes inseparably linked if any two-cell bounded by one of them always contains at least one point of the other curve in its interior (Fig. 3).

The essential part of the knottedness of a curve consists of a linking of two branches of the curve (Fig. 4). Since this is so, some of the properties of knots can be deduced from a study of two linked curves, though it is obvious that the presence of such a link does not insure that the curve is knotted (Fig. 1).

2. Dimensionality and Linking in $S_3$ and $S_4$.

THEOREM. A knot, or link, consisting of a closed curve, or set of closed curves, can always be resolved in $S_4$ into a circle, or set of unlinked circles.

Consider two planes in $S_3$, (Fig. 5), intersecting in a line $\chi$. Let a circle* lie in each, with the centers on $\chi$, but the two centers not at the same point, such that the distance between the centers is less than the sum of the radii and greater than the difference, which insures linking.

Consider the system as immersed in $S_4$. Choose a point P on $\chi$ but not inside both circles. The two planes can be separated

*The proof will apply, with slight modifications, if instead of circles we have any simply closed curves. In fact, the metric ideas are entirely unessential to the proof. They are used here merely to simplify the language. In this and in following proofs where metric ideas are introduced, only properties of intersections of varieties immersed in Euclidean number spaces will be used, and in spite of any such freedom of expression, it is to be understood that we are looking at the problem from a combinatorial point of view, and that any of the statements made can surely be translated into the less expressive language of pure combinatorial analysis situs.

except at the point P, by rotating one of the planes about an axis through P, which may be taken as a line in the $S_3$, out of the $S_3$ in which the two planes lie. During the rotation no point of one plane will be in contact with the other except P. Consequently there is no longer any common point on the 2-cell which each curve of the link in its plane bounds, and they are unlinked. The line $\chi$ is replaced by two lines, $\chi_1$ and $\chi_2$, one in each plane, which intersect at P. Now if we move one of the curves, $C_1$, which does not contain P in its interior, in its plane until it no longer contains any point of the line $\chi_1$, then join the planes a second time by reversing the rotation, until $\chi_1$ and $\chi_2$ again coincide in $\chi$, we shall have returned the system to $S_3$, but the two circles are now unlinked.

The last paragraph may be more easily understood, perhaps, from an analogous situation in $S_3$. Consider a plane $S_2$ and a line $\chi$ in it (Fig. 6). (Because of the dimensionality we cannot use two varieties of the same dimension as we did before.) Immerse the system in $S_3$. If we fix a point P on the line, the line can be rotated out of the plane about an axis through P, which can be any line through P (except the line $\chi$ or the perpendicular to the plane at P). At any stage of the rotation no point of the line except P is in contact with the plane. Now suppose a circle lying in the plane, and a point Q interior to it. Draw $\chi$ through Q, cutting the circle at R and R'. Rotate the line about some point P on it, not Q, into $S_3$. Slide Q along the line away from P until it passes R or R'. If we return the line to the plane by reversing the rotation, the point Q which was interior to the circle is now outside it.

The process described above will serve by repeated application, to unknot any knot in $S_3$, for the transformations are so

defined that bending and stretching are permitted. The essential part of the knotting is the linking of two branches. If we join two points of each branch by a line so that the resultant closed curves are inseparably linked (Fig. 7), and then deform this portion of each curve until it lies in a plane, the two planes will intersect in a line $\chi$. If we bend this part of the configuration into $S_4$, we can unlink the branches as above. Since the knot is flexible and extensible, it is possible to do this without disturbing the rest of the configuration.

Metric considerations may be avoided by the following procedure. Consider a simple closed curve in each plane so placed that the line of intersection of the two planes, $\chi$, cuts each curve in two points and, therefore, contains a segment of points interior to each curve, and placed so that on $\chi$ there are two segments, each of which contains only points interior to and on one curve, and one segment containing points common to both, as in Figure 8. The proof then proceeds as above.

3. The Knotted Variety. A knotted variety of n-dimensions, or n-knot, $K_n$, will be defined as a non-singular closed manifold which is homeomorphic to, but not isotopic to, the boundary of an $(n + 1)$-dimensional sphere. Hereafter in this paper, the boundary of an $(n + 1)$-dimensional sphere will be referred to as an n-sphere, defined as in Veblen (1) page 44. The n-knot consists, then of a finite number of 0, 1, $\cdots$, n-dimensional cells, so disposed that the incidence relations are those of an n-sphere. The n-sphere when it occurs, will be considered as the degenerate case. Any variety equivalent (see below) to it will be called unknotted.

Two n-knots, $n > 1$, are equivalent or of the same type if one can be reduced to the other by a set of a finite number of

the following transformations. For $n = 1$, see Section 1.

(i) To subdivide an i-cell into two i-cells $2 \leqq i \leqq n$, by creating a new $(i - 1)$-cell interior to it, bounded by an $(i - 2)$-circuit on the boundary of the original i-cell.

(ii) To reverse the last process, that is, to amalgamate a pair of adjacent i-cells along the $(i - 1)$-cell which is their common boundary.

(iii) To change the shape of a knot by continuously displacing an i-face in such a manner that the i-face never acquires a singularity during the process. In other words, an i-cell, $E_i$, together with the star of $i + 1, \cdots$, n-cells of which it is the center, may be replaced by any other i-cell, $E_i'$, and a star of $i + 1, \cdots$, n-cells having the same incidence relations as the original star, and the same boundary, provided $E_i'$ has no point in common with the knot, and that an $(n + 1)$-cell can be found, bounded by the two stars and their common boundary, which does not contain any point of the knot.

The diagram of a $K_n$ is the projection of the knot from a conveniently chosen $S_{n-1}$ of the $S_{2n+1}$ in which the knot is immersed (see below) onto an $S_{n+1}$. The diagram will be a closed orientable n-dimensional variety, having, in general, singularities, which, however, by resorting to one or more of the operations described above, may be made into double $V_{n-1}$'s with distinct tangent $V_n$'s at all their interior points. These singularities are the crossings of the diagram. There will be singularities consisting of cells of lower dimensions at the intersections, and only at the intersections, of the singularities of dimension $n - 1$. The $S_{n+1}$ of the diagram will be divided into a finite number of regions, the regions of the diagram.

A diagram will be called reduced when, after a series of applications of transformations (i) and (ii), there are no $(n-1)$-cells other than the crossings.

We shall study some of the properties of knotted varieties by investigating the properties of linked varieties. Two knotted varieties will be called linked or inseparably linked if a cell bounded by one of them always contains at least one point interior to the other.

4. Dimension of $K_n$.

THEOREM. Two simply closed varieties of dimension n can be inseparably linked in $S_{2n+1}$.

For our purposes it is sufficient to consider two n-dimensional spheres in $S_{2n+1}$. Choose a set of rectangular axes $x_1, \cdots, x_{2n+1}$. Let the spheres be

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = r^2$$
$$x_{n+2} = \cdots = x_{2n+1} = 0,$$

and

$$(x_1 - a)^2 + x_{n+2}^2 + x_{n+3}^2 + \cdots + x_{2n+1}^2 = r^2$$
$$x_2 = x_3 = \cdots = x_{n+1} = 0 \qquad \text{(Fig. 9)}.$$

If we choose a so that $a \leq r < 2a$, the two spheres are linked but do not intersect. Keeping r fixed, let a increase until $a = 2r$. Then the spheres have a common point, $(r, 0, \cdots, 0)$, where the last 2n coordinates are zero. As a approaches 2r the spheres are linked. At $a = 2r$ they have a point in common, and as a becomes greater than 2r they are unlinked. Hence as a increases continuously, the first sphere passes through the second at the point $(r, 0, \cdots, 0)$, and intersects it there. From the symmetry of the figures it is obviously not possible to move one of the spheres outside of a hypersphere containing the other without

their having at least one common point at some stage of the transformation, that is, the two spheres are inseparably linked. Whence it follows that any two closed varieties, each of n-dimensions, and having a part whose incidence relations are those of n-spheres, may be inseparably linked in $S_{2n+1}$.

COROLLARY. Two closed varieties of dimension $n' < n$ cannot be inseparably linked in $S_{2n+1}$.

For if two spheres are of dimension $n' < n$, we can write their equations so that they have no common variables, and therefore the spheres can be immersed in distinct subspaces of $S_{2n+1}$. The two subspaces can be moved apart to an arbitrary distance without intersecting. This corollary may also be proved by a simple generalization of the proof of the theorem in Section 2. The same argument gives the following

COROLLARY. Two linked varieties, each of dimension n, can be unlinked in $S_{2n+2}$.

It is also true that two varieties of dimension n', with $n \leqq n' \leqq 2n + 1$ can be inseparably linked in $S_{2n+1}$. In fact, from the proof of the theorem in Section 2, it is clear that two crown shaped surfaces, or two closed surfaces like the surface of the torus, or even two solid rings, can be linked in $S_3$, but only such linked varieties as can be isotopically deformed so that each component of the link lies in a plane can be unlinked in $S_4$. In this paper we consider only the variety of least dimension which can be properly knotted in the space under consideration. A link consisting of two closed surfaces or two solid rings cannot be unlinked in $S_4$, for a proof similar to the above shows that two closed surfaces in $S_4$ in a linked position, in undergoing a deformation in which one leaves a hypersphere containing the other, necessarily have a finite number, greater

than zero, of points in common at some stage of the deformation. Hence they cannot be unlinked in a space of less than five dimensions. This leads to the following

COROLLARY. In $S_{2n+1}$ two varieties of dimension 1, 2, ···, 2n + 1 can be linked, that is, two non-intersecting closed varieties can be so placed that one of them cannot be moved arbitrarily without intersecting the other.

For varieties of dimension m, $m \geqq n$, they are inseparably linked, for $m < n$ they are not inseparably linked.

5. Existence of the Knotted Variety. To prove that a $K_n$ can actually be found in $S_{2n+1}$ we shall show how to construct a $K_2$ in $S_5$ in a manner entirely analogous to the construction of the trefoil knot in $S_3$. The generalization to n-dimensions is immediate.

Consider the diagram of the trefoil knot (Fig. 11), as consisting of two one-circuits, each containing three 1-cells, a, b, c and $a^1$, $b^1$, $c^1$ respectively, and the three points $C_1$, $C_2$, $C_3$, which are, in the diagram (but not in the knot) common to both. Let the circuits have no other common point. By relettering the one-cells if necessary, let the 2-cells bounded by $a^1$, $b^1$, $c^1$ contain all the points interior to the 2-cells bounded by a, b, c. Now let the curve of the diagram be immersed in $S_3$, and let $s_1$, $s_2$, $s_3$ be one-cells formed by amalgamating $ab^1$, $ca^1$ and $bc^1$. Consider the knot as projected from some point in $S_3$. Then at the double points, by referring to the knot from which the diagram was obtained, we can distinguish the branches. At each double point one branch will lie nearer to the center of projection than the other does. By introducing a suitable notion of distance, measured from the center of projection, we can speak of the upper and lower branches as being those nearer to and farther from the

center of projection. In this particular diagram, let $s_1$, $s_2$, $s_3$ pass over the branches they cross, making the boundary points of these lines lie under the other branches. In the figure, if the diagram is oriented as indicated by the arrow, dots are placed in the regions on the left side of the branch which passes under the one it meets. That the variety resulting from this construction is the trefoil knot is obvious. This fact will also be shown, however, by a consideration of its group, which is that of the trefoil knot.

To construct the diagram of the simplest knot in $S_5$, consider the diagram as consisting of two 2-dimensional spheres, each consisting of three 2-cells, a, b, c and $a^1$, $b^1$, $c^1$ (Fig. 12), and three 1-cells, $C_1$, $C_2$, $C_3$ which are in the diagram common to both, and let the spheres have no other common 0- or 1-cells. By relettering the 2-cells if necessary, let the 3-cell bounded by $a^1$, $b^1$, $c^1$ contain all points interior to the 3-cell bounded by a, b, c. Now let the surface of the diagram be immersed in $S_5$, and let $s_1$, $s_2$, $s_3$ be 2-cells formed by amalgamating $ab^1$, $ca^1$ and $bc^1$ respectively. Consider the knot as the projection of a surface from some line chosen in a sufficiently general portion in $S_5$. The projection is effected by joining every point of the diagram to the line by a plane. Then at the double lines it is possible to separate the branches, and let one lie "under" the other, as will be explained in the next paragraph. Let $s_1$, $s_2$, $s_3$ pass "over" the branches they cross, making the boundaries of these cells lie "under" the other branches. That the surface thus formed is not deformable into a sphere and is therefore knotted, will be shown by setting up its group, which is not simply isomorphic to that of the sphere.

Note on "over" and "under" as applied to two surfaces in $S_5$. Given two arbitrary lines, not necessarily straight, in $S_3$, as viewed from a general point in $S_3$ they appear to intersect in a finite number of points or line segments. At these apparent intersections it is possible to distinguish the two curves, one as lying over (or nearer) and one under the other. This relation is, however, characteristic only of the particular point from which the lines are viewed or projected, and is valid only in a restricted neighborhood of the intersection, but it serves effectively to distinguish the branches for our purposes. The distinction obviously holds, since the curves are connected and continuous, for sufficiently small displacements of the center of projection and also of the curves themselves.

To project a $K_2$ in $S_5$ into an $S_3$, if the result is to be a two-dimensional variety, it is necessary to use a line $\chi$ as center of projection, and planes as projectors. We can always choose $\chi$ in a sufficiently general position so that the projection will have only double lines (curves) with distinct tangent planes, and their intersections, as singularities, and also so that the complete projection can be enclosed in a 3-cell in a finite part of $S_3$. Each double line is the projection of two non-intersecting curves $C_1$ and $C_2$ of $K_2$. No singularities of higher dimension occur.

In $S_5$ a projecting plane connecting a point of the diagram to the line $\chi$ will in general not contain any other point of the diagram. The planes determined by the double lines of the diagram, however, contain double points, and we shall separate the surface along these points as follows. Consider a double line. The projecting planes determined by its points all pass through $\chi$, and hence have no other points in common. On each plane there is a double point P of the double curve C. If we

choose a point Q on $\chi$, we can connect each double point to it by a line lying in the projecting plane determined by that point. Now we can separate the double points by sliding them apart along these lines with the result that the point P takes the two positions indicated by P' and P". Then the distance from Q of one point, P', of the pair corresponding to any double point P, will be less than that of the other, P". As we slide the points apart in their planes, two parts of one sheet through the double line will be incident to each, and because of the continuity of $K_2$ as a point set, the direction of the separation of adjoining double points in neighboring projecting planes will be well determined and in the same direction for all points of one sheet, up to the multiple points (0-cells) of the diagram. Hence, if a point in the now separated double line on one sheet is at a lesser distance from Q than the double line in the other, we say the sheet of the $K_2$ containing this line passes "over" the other sheet at the double line. We can separate all the double lines of the diagram in the same manner. Each of the original double lines is now replaced by two lines each having the end points of the original line. Each is on a different sheet, and the surface no longer contains multiple lines.

The separation of the multiple points is induced by the separation of the double lines. For example, at one of the multiple points of Figure 12, suppose the double lines have been divided into $\overline{C}_1$, $\underline{C}_1$, $\overline{C}_2$, $\underline{C}_2$, $\overline{C}_3$, $\underline{C}_3$, where $\overline{C}$ means the upper line at C, and $\underline{C}$ the lower line at C, (Fig. 13), and that the end points are for the moment disjoined. From the construction of the knot, the end $\overline{C}_1$ is incident to the end $\underline{C}_2$ and $\underline{C}_3$, $\overline{C}_2$ to $\underline{C}_3$ and $\underline{C}_1$, $\overline{C}_3$ to $\underline{C}_1$ and $\underline{C}_2$. If we make these connections by one-cells, the one-cells form a circuit. This circuit in $S_5$ is

unknotted, that is, bounds a 2-cell, which is incident to the 2-cells a, b, c, a', b', c'. When the other multiple point has been treated in the same manner, all singularities have been eliminated from the $K_2$. The $K_2$ is now easily seen to be homeomorphic to a sphere by consideration of the incidence relations of the 1-cells and 2-cells.

6. The Group of $K_n$ in $S_{2n+1}$. The knot group of $K_1$, originally defined by Dehn, and somewhat simplified by Alexander (1) page 289, whom we follow here, is merely the ordinary topological group of the space S exterior to the knot. Let us fix upon some point P of the space S, such as the point from which the knot is projected to obtain its diagram. Then in the space S, each closed, sensed curve beginning and ending at P determines an operation of the group. Moreover, two different sensed curves determine the same operation, if, and only if, one may be deformed continuously into the other within the space S, while its ends remain fixed at the point P. During the deformation the curve may cut through itself at will, but it must never come into contact with the knot, as that would involve its leaving the space S. The condition that a curve determine the identical operation is that it be continuously deformable into the point P itself. If two sensed curves $C_1$ and $C_2$ correspond respectively to the operations $r_1$ and $r_2$ of the group, the sensed curve $C_1 + C_2$ obtained by joining the initial point of $C_2$ to the terminal point of $C_1$ determines the operation $r_1 + r_2$.

The group of a knot may be obtained, at once, from a diagram of the knot. Let us flatten out the knot until it coincides, sensibly, with the curve of its diagram. Then, if we pick out at random a region $r_0$ of the diagram, there will be one generator of the group corresponding to each of the other $\nu + 1$ regions

$r_j$, where the generator in question is the operation determined by a curve which starts from the point P, crosses the region $r_j$, passes behind the plane of the diagram and returns to the point P by way of the region $r_0$. It is easy to see, by inspection, that to each crossing point of the diagram there corresponds an identical relation of the form

$$r_j - r_k + r_\chi - r_m = 0,$$

(or in Dehn's notation $r_j r_k^{-1} r_\chi r_m^{-1} = 1$), where if the symbol $r_0$ appears in the relation it must be set equal to zero (in Dehn's notation, equal to one).

In considering a group for a $K_n$ in $S_{2n+1}$, $n > 1$, we cannot use Dehn's method of considering the fundamental path-group of the space in which the knot is immersed, with the knot (and a $(2n + 1)$-dimensional neighborhood of each point of it) removed, since obviously by the first corollary to the theorem in Section 4, if a closed variety of dimension $n + 1$ is removed from $S_{2n+1}$, any closed one-dimensional circuit can still be deformed into any other. That is, there is no actual looping or linking that cannot be readily removed by admissible deformations. Consequently the path group of the $S_{2n+1}$ with $K_n$ removed is more logically constructed as the group whose operations are determined by all possible n-spheres in the $S_{2n+1}$ with $K_n$ removed, having a given element (a point will suffice) in common, where two n-spheres which can be deformed into the same n-sphere by the admissible deformations determine the same operation.*

*A similar situation in the case of homeomorphisms arises, and has been frequently discussed, in the duality theorems for r and n - r - 1 dimensional Betti-groups, etc.

The path group can then be set up immediately as in Dehn (1), a group with one generator for each region of the diagram, and one relation for each crossing. To do this we make use of the relation of upper and lower branches established in Section 5, which makes the determination of the generating relations equivalent to the process described by Dehn. For example, the group of the knot whose diagram is given in Figure 12, is simply isomorphic to the group of the trefoil knot of Figure 11. It has three relations, $S_1$, $S_2$, $S_3$, one for each crossing, and five generators, $r_0$, $r_1$, $\cdots$, $r_4$, one for each region of the diagram. There are always four and only four regions incident to any one crossing. For example, consider $C_1$ in the diagram in Figure 12. If $c^1b$ lies under $cb^1$, a plane cross section perpendicular to $C_2$ will be as in Figure 14. If $r_1$ and $r_4$ as well as $r_2$ and $r_3$ are adjacent along the under sheet, and the four regions are taken in the order indicated in the figure, we have the relation

$$S(C) = r_1 - r_2 + r_3 - r_4 = 0.$$

The three relations for this knot are:

$$S(C_1) = r_2 - r_4 + r_1 - r_0 = 0,$$
$$S(C_2) = r_0 - r_3 + r_4 - r_2 = 0,$$
$$S(C_3) = r_0 - r_1 + r_4 - r_3 = 0.$$

This takes the form of Dehn's group if we set $r_0 = 0$. It becomes Alexander's variation of Dehn's group if we replace $r_4$ by a suitable one of its equals, $r_1 + r_2 = r_2 + r_3 = r_3 + r_1 = r_4$, and also replace $r_0$ by 0. A different replacement of these same values of $r_4$ gives Reidemeister's group, Reidemeister (1) page 44.

Since the inverse of a relation is a relation, and a cyclic permutation of the elements of a relation is a relation,

the group is unambiguously determined when we know which sheets pass under those met by them at the double lines.

7. Construction of $K_n$ by Empirical Methods. Tait, with the assistance of Little and Kirkman, constructed all the known types of knots, $K_1$, up to eleven crossings, by empirical methods. The process, described briefly in Tait (1), is essentially as follows:

Any closed plane curve which has double points only, may be looked upon as the projection of a knot in which each branch of the curve passes alternately under and over the successive branches it meets. Go continuously around the projection of the knot, putting A, B, C, etc. at the first, third, fifth, etc. crossings you pass, until you have put letters to all. Then go around again, putting down the name of each crossing in the order in which you reach it. The list will consist of each letter employed taken twice over. A, B, C, etc. will occupy, in order, the first, third, fifth, etc., places; but the way in which these letters occur in the even places fully characterizes the drawing of the projected knot. To reconstruct the diagram (Fig. 15)

EFBACD = AEBFCBDAECFD/A etc.,

choose two apparitions of the same letter between which no other letter appears twice, thus, AECFD/A forms such a set. It must form a loop of the curve. Draw such a loop, putting A at the point where the ends cross, and the other letters in order, either way, around the loop. Fill in the rest of the cycle the same way. Now we need to know which branch passes over the other at each crossing. This can be supplied by a + or - sign attached to each letter where it occurs in the statement of the order in the even places. Furnished with this process, we find that it becomes

a mere question of skilled labor to draw all the possible knots having any designated number of crossings. The requisite labor increases with extreme rapidity as the number of crossings is increased, because of the appearance of arrangements incompatible with the properties of a self-intersecting plane curve, and of numerous distortions of one and the same knot.

Tables giving knotted $K_1$'s up to and including nine crossings can be found in Alexander (1) and Reidemeister (1). To construct knotted $K_2$'s in $S_5$, we can begin with the diagram in $S_3$. To each knot $K_1$ we can find a corresponding knot $K_2$ in $S_5$ by the simple device of considering the pairs of 0-cells, which bound 1-cells in a reduced diagram of $K_1$, as circuits. To each 0-circuit in the diagram of a $K_1$ we construct a corresponding 1-circuit in $S_3$ for the $K_2$. For example, consider the $K_1$ of four crossings (Fig. 16). To construct the corresponding $K_2$, draw a circuit containing the 1-cells $b_1^1$ and $b_2^1$ and the 0-cells $b_1^0$, $b_2^0$. On this 1-circuit construct the 2-cells $b_1^2$, $b_2^2$ as in the Figure 17, to make a sphere. These correspond to $a_1^1$ and $a_2^1$ in Figure 16. Now there is another circuit $a_1^0$ and $a_3^0$ which has a half circuit in common with $a_1^0$ and $a_2^0$. Construct its correspondent $b_1^1$ and $b_3^1$ and let these two 1-cells bound a 2-cell $b_3^2$; similarly for the other circuits of $K_1$. This construction forces the diagram to have only two 0-zero cells. The crossings at these 0-cells can be made as in Section 5. The crossings on the surface are to be made as on the $K_1$. The surface is orientable, and its group is simply isomorphic to the group of the $K_2$ from which it is derived.

Composite $K_2$'s can be constructed in various ways by setting up sets of one-circuits on more than one pair of points.

8. Alexander's Polynomial Invariants. In 1928 Alexander announced the discovery of a simple and effective knot invariant which takes the form of a polynomial, $\triangle(x)$, with integer coefficients, where both the degree of the polynomial and the values of its coefficients are functions of the curve with which it is associated, Alexander (1). These invariants are intimately associated with the knot group defined by Dehn and described in Section 6 of this paper. To extend the theory to $K_n$ it is necessary to extend the definitions of signature and index.

For $K_2$, since the surface is defined to be orientable, we can, in the diagram, choose one side to be the inside and call the other the outside. (In $S_5$ this distinction is illusory since the $K_2$ does not inclose any space.) The choice is purely arbitrary, but once defined on any two-cell, is well determined all over the diagram, and the aforesaid relation corresponds to the relation of right and left side of the curve of the diagram of a $K_2$.

At each crossing of the diagram of a $K_2$, two of the four corners (dihedral angles) will be marked with a line, used to indicate which sheet passes over and which under. The lines will be drawn in the corners, roughly paralleling the crossings, on the outside near the sheet which passes under the other. This convention is such that an insect crawling on the outside of the surface, always has the lines above him, one just before and one just after he crosses a double line when he is travelling on the under sheet. Two corners, both containing or not containing lines, are said to be of like signature. If one is lined and the other is not, they are of unlike signature.

Consider the diagram as reduced so that it consists only of 2-cells bounded by the double lines. Let these two-cells be

consistently oriented, that is, so oriented that any two 2-cells incident along a one-cell form, together with their common one-cell, a new oriented 2-cell. Let the sheets be connected at the crossing lines so that one sheet actually passes through the other at every crossing. Four 2-cells meet at every one-cell, and form there part of the boundaries of four regions incident to the one-cell, (Fig. 18). The one-cell will have the same orientation in two of the 2-cells and the opposite orientation in the other two. In either sheet the one-cell appears with one orientation in one 2-cell and the opposite orientation in the other. Of the four regions, two which are placed vertically opposite, will have the property that the two-cells bounding it at the corners meet at the double line with the same orientation. In Figure 18 these are A and C.

Assign to any one region an integer p which will be called the index of the region. To assign an integer to any other region, let it be so chosen that in crossing from one region to another through a two-cell, if the crossing is made from outside to inside the index decreases by one. This determines without contradiction the indices of all the other regions. In the figure the lined regions are B and C, the indices of A, B, C, D are p, p + 1, p, p - 1 respectively. It is easy to verify that at every crossing there will be two opposite corners of index p, and two of indices p + 1 and p - 1 respectively. The index p which occurs twice will be referred to as the index of the crossing.

There can be only one kind of crossing on any given knot $K_2$, for the only arbitrary choice is that of which sheet is over and which is under. Then if the 2-cells in both sheets are oriented, and the dotted corners are known, the outside of both sheets is well determined.

There are, however, two types of crossings, corresponding to the two possible orientations of a 2-cell (Figs. 18 and 19). They cannot both occur on the same knot since the orientation of the 2-cells in the first is inconsistent with that in the other. An analogous situation will arise in all $K_n$'s where n is even.

Nevertheless, we can distinguish two types of crossings on a $K_2$, if we consider all the one-cells of the reduced diagram as beginning at one of the two singular zero-cells (Section 7) and ending at the other. Then all the double lines are oriented. We can consider a crossing as right handed or left handed according as the under sheet can be brought into coincidence with the upper sheet by a rotation of less than two right angles corresponding to that of a right-handed or left-handed coordinate system having the oriented one-cell as one axis, the second axis lying in the lower sheet, the third in the upper sheet, with positive directions in the 2-cells bounding the region of index p + 1.

At either kind of crossing the two unlined corners are of indices p - 1 and p, respectively. The two lined corners are of index p + 1 and p. At a right-handed crossing the lined corners of index p preceeds the lined corner of index p + 1 as we rotate around the double line positively in a right-handed system, whereas in the left-handed crossing the former follows the latter. As in the case of $K_1$, at a crossing line the two corners of index p may belong to the same region of the diagram. We notice that on the boundary of a region of index p, only crossing lines of index p - 1, p and p + 1 may occur. Finally we recall that the whole system is determined to within an additive constant only, since the index of some one region or crossing has to be assigned before the indexing of the figure as a whole becomes determinate.

The equations of the diagram can now be written immediately from the diagram. If a diagram has two 0-cells and $\nu$ crossing lines (1-cells), it has $2\nu$ two-cells and by Euler's formula for polyhedra, $\nu + 2$ regions. Now the results of Alexander's paper follow immediately for all knotted surfaces whose diagrams are derived from those of $K_1$'s.

We give here an example of a trefoil, $K_2$, (Fig. 20), derived from a trefoil $K_1$ (Fig. 21). Corresponding to each crossing, $c_i$, of the diagram, we write a linear equation. If the four regions adjacent to the crossings are $r_j$, $r_k$, $r_\chi$ and $r_m$, as we go around the crossing in the positive sense, and if $r_j$ and $r_k$ are the lined crossings, we write the equation as follows:

$$c_i(r) = xr_j - xr_k + r_\chi - r_m = 0.$$

The $\nu$ equations determined by the $\nu$ crossings are the <u>equations of the diagram</u>. The cyclic order of the elements in these equations plays an essential role and is not to be disturbed. The coefficients x indicate which corners are lined. The equations of the diagram of the trefoil in Figure 20 are as follows:

$$\begin{aligned} c_1(r) &= xr_3 - xr_1 + r_4 - r_5 = 0, \\ c_2(r) &= xr_3 - xr_2 + r_4 - r_1 = 0, \\ c_3(r) &= xr_3 - xr_5 + r_4 - r_2 = 0. \end{aligned}$$

To obtain the invariant polynomial, we treat the equations of the diagram as a set of ordinary linear equations in which the ordering of the terms is immaterial.

The matrix of the coefficients is a rectangular array of rows and $\nu + 2$ columns, one row corresponding to each crossing, and one column to each region of the diagram. Alexander has proved that this matrix has the property:

If the matrix M is reduced to a square matrix $M_0$ by striking off two of its columns corresponding to regions with consecutive indices p and p + 1, the determinant of the residual matrix $M_0$ will be independent of the two columns struck out, to within a factor $\pm x^n$.

The polynomial $\Delta(x)$ which is the determinant of a square matrix obtained by striking out two rows of consecutive indices, when divided by a term $\pm x^n$ chosen in such a way as to make the term of lowest degree a positive constant, is a knot invariant.

In the case of the trefoil, the invariant is

$$\Delta(x) = 1 - x + x^2.$$

For, from the equations of the diagram we have the matrix

$$\begin{Vmatrix} -x & 0 & x & +1 & -1 \\ -1 & -x & x & +1 & 0 \\ 0 & -1 & x & +1 & -x \end{Vmatrix}.$$

If we assign the indices in such a way that $r_1$ is of index 1, then $r_2$ and $r_5$ are of index 1, $r_3$ of index 2, and $r_4$ of index 0 as indicated by the superscripts in the figure. We assign these indices to the corresponding columns of the matrix. The determinant obtained by striking out the last two columns, of indices 0 and 1 respectively, is

$$\Delta_{01}(x) = x(1 - x + x^2).$$

The determinant obtained by striking out the second and third columns is

$$\Delta_{12}(x) = 1 - x + x^2.$$

The invariant for this trefoil is, of course,

$$\triangle(x) = 1 - x + x^2.$$

Two matrices are said to be $\epsilon$-equivalent if it is possible to transform one into the other by means of the operations

(i) multiplication of a row or column by - 1.

(ii) interchange of two rows (or columns).

(iii) addition of one row or column to another.

(iv) bordering the matrix by one new row and one new column, where the element common to the new row and column is 1, and the remaining elements of the new row and column zeros; or the inverse operation of striking out a row and column of the type just described.

(v) multiplication or division of a row or column by x.

Two polynomials will be said to be $\epsilon$-equivalent if they differ, at most, by a factor of the form $\pm x^n$.

It is now necessary to show that: If two diagrams represent knots of the same type (are equivalent), their matrices are $\epsilon$-equivalent. When a knot is deformed, the equations of the diagram remain invariant so long as the topological structure of the diagram does not change. A change in the structure of the diagram may occur in one of the following ways:

(A) The surface may acquire a loop and a crossing, or lose a loop and a crossing by a deformation of the inverse sort (Fig. 23). Since the crossing bounded two 2-cells of the surface before the deformation, it must pass through the multiple points on the surface. The effect on the diagram of the $K_2$ is to introduce (or remove) a new 2-cell bounded by the new crossing line considered twice, which is the same as the effect on the diagram of the $K_1$, that is, to introduce a new one-cell bounded by the crossing point considered twice (Fig. 23).

(B) One sheet may pass under another with the creation of two new crossing lines, provided that the corners are lined so as to imply that the lower branch at one is also the lower branch at the other (Fig. 24).

(C) If there is a three-cornered region of the diagram, bounded by three 2-cellular regions and three crossing lines, and if the sheet corresponding to one of the three 2-cells passes beneath the sheets corresponding to the other two, then any one of the three sheets may be deformed past the crossing line formed by the intersection of the other two (Fig. 25).

Any variation in the structure of the diagram which can be derived from a deformation in a knot which it represents, can be compounded of variations of these types. Equivalent sets of deformations, easily generalized to $K_n$, can be found in Alexander (2) and Reidemeister (1) page 7. Alexander has shown that the polynomial $\Delta(x)$ obtained from the matrix of the equations of the diagram before the deformations is $\epsilon$-equivalent to that obtained after the deformations. Consequently, for any $K_2$ whose projection in $S_3$ has two multiple points only, an invariant polynomial $\Delta(x)$ can be found, and it is an invariant of the knot.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

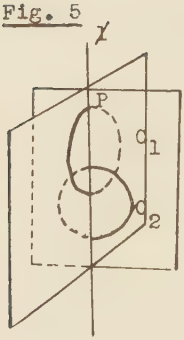



Fig. 6

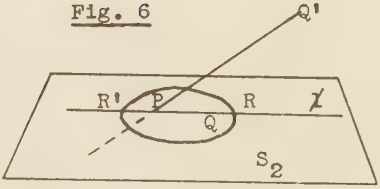



Fig. 7

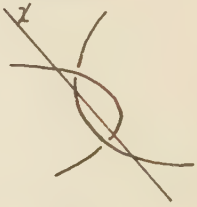



Fig. 9

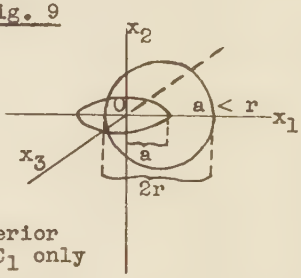



Fig. 10

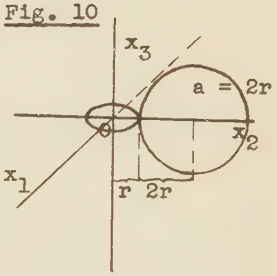



Fig. 8

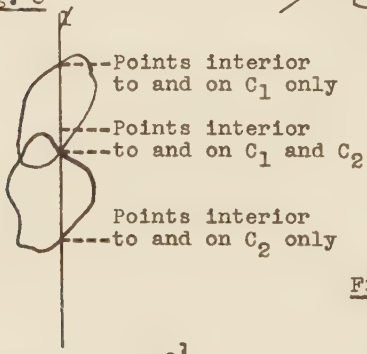



Fig. 12

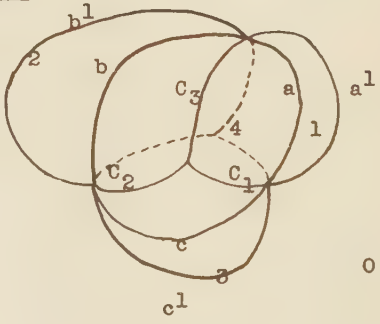



Fig. 11

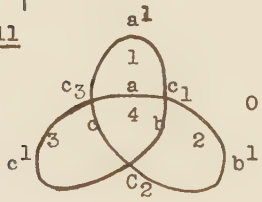



Fig. 13

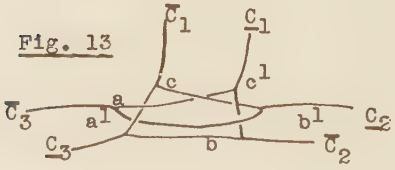


Fig. 14

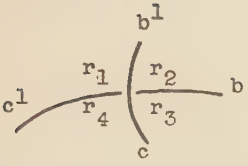



Fig. 15

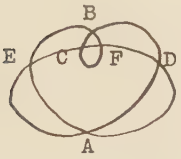



Fig. 16

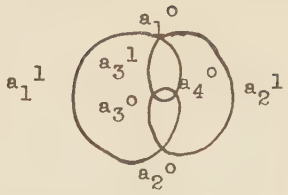



Fig. 17

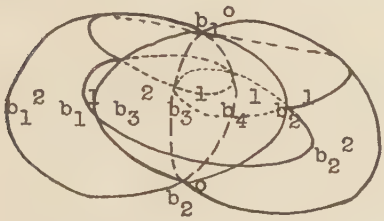



Fig. 18

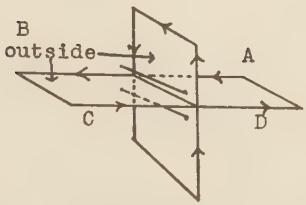



Fig. 19

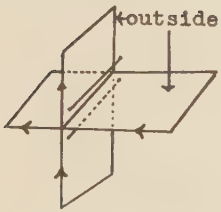



Fig. 20

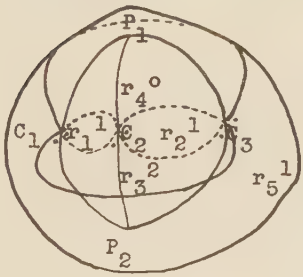



Fig. 21

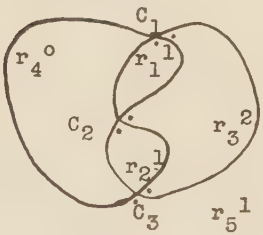
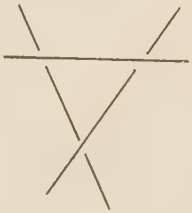



Fig. 22

Fig. 23

Fig. 24

Fig. 25



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VITA

Anna Adelaide Stafford was born in Chicago, Illinois, on August 20, 1905. She received her primary and secondary education in the public schools of Chicago; Marshfield, Wisconsin; Barnum, Minnesota; and Saint Louis, Missouri. In 1922 she entered the Western College for Women at Oxford, Ohio, and was granted the degree of Bachelor of Arts in 1926. She taught mathematics and science at St. John Baptist School, Mendham, New Jersey from 1926 to 1928 and 1929 to 1931, and at Darlington Seminary, West Chester, Pennsylvania, in 1928-1929. She attended the University of Chicago in the summers of 1927, 1928, 1929 and 1931. In the fall of 1931 she entered upon a period of two years of residence, receiving the degree of Master of Science in December 1931. It has been her privilege to study under Professors Albert, Barnard, Bell, Bliss, Coble, Dickson, Graustein, Lane, Logsdon, Lunn, MacMillan, Mitchell and Slaught.

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